Time-Dependent Probabilistic Failure of Coated Components

Brice N. Cassenti*
United Technologies Research Center, East Hartford, Connecticut 06108

A probabilistic time-dependent failure model has been developed for the failure of materials. The model is an extension of a previously developed theory for static probabilistic failure that includes fatigue and creep rupture failure. The model has been applied to coatings and ceramic matrix composities. It has been shown to predict accurately the segment size in coatings and the statistical failure locations in brittle tensile specimen. Many of the features of the model can be illustrated using one-dimensional (i.e., through thickness) analyses of layered materials. Such analyses have direct applications to coated components. The model has been incorporated in a computer program that processes output from a finite element simulation and, hence, can be applied to structural analyses, with arbitrary loading, in any number of dimensions.

Introduction

HOT section gas turbine components are becoming more dependent on ceramic materials to satisfy ever increasing thermal requirements. Thermal barrier coatings and ceramic combustor liners are examples of the uses for high temperature ceramic materials. Although such materials can withstand extremely high temperatures, they are brittle and, hence, possess considerable scatter in their strength. At United Technologies Research Center (UTRC) a probabilistic model has been developed for the failure of materials. The model includes static, creep, and fatigue failure. The static model was developed previously at UTRC[1] and has been applied to the analysis of composite material components,[1] solid rocket motor cases,[2] and the prediction of failure locations in brittle specimen tests.[3]

The model is based on the weakest link failure theory of Weibull.[4,5] Such models make direct use of statistical failure data and, hence, convert deterministic failure analyses into more realistic probabilistic failure analyses. Weibull's original theory predicts only the scatter in static failure loads, and does not predict either the scatter in failure location or failure in time-dependent situations such as fatigue or creep rupture. Deterministic models for time-dependent failure have been the subject of a significant amount of research including fracture, crack propagation, and fatigue. More recently, damage based models[6] have also been introduced. Much of this previous research can be cast directly into a probabilistic form. In fact, fracture mechanics has been viewed from the context of statistical formulations[7-9] and can be used as a basis for more realistic models. Life prediction models for coated components is currently the subject of investigation[10] and will be the major focus of this paper.

Review of Theory

The theory described in Ref. 1 begins by assuming that the probability of failure δf occurring in an infinitesimal volume δV is given by

$$\delta f = \psi \delta V \tag{1}$$

The parameter ψ is taken to be a function of the stress σ_i. This dependence can be implicit, e.g.,

$$G(\sigma_i, \psi) = 0, \qquad i = 1,2,\ldots,6 \tag{2}$$

Equation (2) is generally empirical and can take on many forms. One that has been used by the author is

$$G(\sigma_i, \psi) = 1 - \sum_{i=1}^{6} \left\langle \frac{\sigma_i}{(\psi V_O)^{1/N_{Ti}} \sigma_{T_i}} \right\rangle^{M_{T_i}} + \left\langle \frac{-\sigma_i}{\sigma_{c_i} (\psi V_o)^{1/N_{ci}}} \right\rangle^{M_{ci}} = 0 \tag{3}$$

where

$$\langle x \rangle = \begin{cases} o & x \leq 0 \\ x & x > 0 \end{cases}$$

is the unit ramp function, $\sigma_{1,2,3}$ are normal stress components, $\sigma_{4,5,6}$ are shear stress components, σ_{T_i}, N_{T_i}, and M_{T_i} are experimentally determined material constants associated with tension failure, σ_{c_i}, N_{c_i}, and M_{c_i} are the equivalent constants for compression failure, and V_o is a reference volume that can be chosen in a convenient manner for a given material but then should remain constant for the material for all other conditions and geometries. Material test data only determine the products

$$\sigma_{T_i} V_o^{1/N_{T_i}} \text{ or } \sigma_{c_i} V_o^{1/N_{c_i}}$$

A more complete discussion of the formulation and interpretation can be found in Ref. 1. Equation (3) can be simplified by considering loading in uniaxial tension only. Then

$$G = 1 - \left[\frac{\sigma_1}{\sigma_{T_1} (\psi V_o)^{1/N_{T_1}}} \right]^{M_{T_1}} = 0 \tag{4}$$

Solving for the failure parameter ψ

$$\psi = \frac{1}{V_o} \left(\frac{\sigma_1}{\sigma_{T_1}} \right)^{N_{T_1}} \tag{5}$$

Equations (1) and, (2) are not enough to specify failure in an actual component, and, hence, additional assumptions are required. The most conservative assumption would be to assume that if any infinitesimal volume δV fails then the whole structure of volume V fails. This assumption forms the basis of weakest link theory. Applying this assumption to Eq. (1) can be shown to yield the following expression for the probability of survival S of a structure, or component, of volume V[1]

$$S = \exp\{ - \int_V \psi(V) \, \mathrm{d}V \} \tag{6}$$

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*Principal Engineer. Senior Member AIAA.

From a consideration of differential forms of the failure law, Eq. (1) can be shown to result in a differential expression for the probability to observe a failure δf° in an infinitesimal volume δV, as[1]

$$\frac{\delta f^\circ}{\delta V} = \sum_{k=1}^{N} \int_{o}^{P_k} S(P') \frac{\partial \psi}{\partial P_k'} \, \mathrm{d}P_k' = \int_{o}^{t} S(t') \frac{\partial \psi}{\partial t'} \, \mathrm{d}t' \tag{7}$$

where P_k' is one of N loads (e.g., pressure, point loads, etc.) acting on the structure, P_k is the maximum value of the load P_k', and $S(P')$ is the probability of survival for the structure with all the loads acting simultaneously. Note that δf° is the infinitesimal probability to observe a failure at a particular time in the loading history, and δf is the infinitesimal probability for failure to occur at the same point in the loading history. If the structure has already failed, δf cannot be observed. For uniaxial tension, with the distribution of Eq. (5), Eq. (6) yields

$$S = \exp\left\{ - \int \left(\frac{\sigma_1}{\sigma_{T_1}} \right)^{N_{T_1}} \frac{\mathrm{d}V}{V_o} \right\} \tag{8}$$

For the case of normal tension loading in the one direction, using Eq. (5), a simple two-parameter Weibull distribution is commonly used. A three-parameter Weibull distribution can also be used[1] but will not be applied in this paper. The integration can be carried over the volume and the surface to include surface flaw sensitivity.[1]

Equation (7) can be integrated over the volume to find the total probability for failure F,

$$F = \int_V \frac{\delta f^\circ}{\delta V} \, \mathrm{d}V \tag{9}$$

It can be shown that the sum of S in Eq. (6) and F in Eq. (9) is 1 (see the Appendix), that is,

$$F + S = 1 \tag{10}$$

The remainder of the paper will extend the previous static failure analysis to time-dependent failure and then will apply the results to coated components. Only volumetric flaws will be considered.

Static Failure Examples

Consider a uniaxial stress σ in a tensile specimen. For a two-parameter Weibull distribution, the failure parameter is

$$\psi(\sigma) = A\sigma^m \tag{11}$$

where A and m are material parameters. The parameter A can be chosen to be

$$A = 1/(V_o \sigma_o^m) \tag{12}$$

where V_o is a characteristic volume and σ_o a characteristic strength.

Recall that the material test data determine the parameter A only. The parameter m in Eq. (11) represents the scatter in failure loads. Higher values of m represent a more deterministic system, whereas lower values of m represent more scatter in the failure loads. If m is infinite, the failure loads would be deterministic. Either one of the parameters V_o or σ_o in Eq. (12) can be chosen arbitrarily, and the remaining parameter can then be determined by Eq. (12). For a uniform tensile specimen, Eq. (7) can be used to find the failure location distribution[1] and results in

$$\frac{\delta f^\circ}{\delta V} = \frac{1 - S}{V} \tag{13}$$

where V is the volume of the specimen and the probability for survival is

$$S = \exp\left[- \left(\frac{\sigma}{\sigma_o} \right)^m \frac{V}{V_o} \right] \tag{14}$$

Integrating Eq. (13) over the volume gives

$$F = \int_v \frac{\delta f^\circ}{\delta V} \, \mathrm{d}V = 1 - \exp\left[- \left(\frac{\sigma}{\sigma_o} \right)^m \frac{V}{V_o} \right] \tag{15}$$

Adding Eq. (14) and (15)

$$S + F = 1 \tag{16}$$

which must hold.

In a similar manner, the uniaxial case of pure bending can be examined. In pure bending the stress is given by

$$\sigma = \sigma_a (2z/h) \tag{17}$$

where σ_a is the stress at the outside surface, h the distances between surfaces, and z the distance from the centerline.

Assuming only tensile stresses cause failure

$$\psi = \begin{cases} \dfrac{1}{V_o} \left(\dfrac{2z\sigma_a}{h\sigma_o} \right)^m & z > 0 \\ 0 & z < 0 \end{cases} \tag{18}$$

from Eqs. (11), (12), and (17).

Substituting Eq. (18) into Eq. (12)

$$S = \exp\left\{ - \frac{V}{2(m+1)V_o} \left(\frac{\sigma_a}{\sigma_o} \right)^m \right\} \tag{19}$$

Comparing Eqs. (14) and (19), a tensile specimen is more likely to fail at the stress σ_a than a pure bend specimen subject to the same maximum stress σ_a. This results from the fact that much less volume is subject to high stresses in the pure bend specimen than in the tensile specimen.

Another common loading situation is that of three point bending. For this case, the stress is given by[1] (see Fig. 1)

$$\sigma = 4\sigma_a \left(\frac{z}{h} \right) H(x) = 4\sigma_a \left(\frac{z}{h} \right) \begin{cases} \dfrac{x}{\ell} & 0 \le x \le \dfrac{\ell}{2} \\ \dfrac{\ell - x}{\ell} & \dfrac{\ell}{2} \le x \le \ell \end{cases} \tag{20}$$

the survival probability is

$$S = \exp\left\{ - \frac{V}{2(m+1)^2 V_o} \left(\frac{\sigma_a}{\sigma_o} \right)^m \right\} \tag{21}$$

and the failure location density is

$$\frac{\delta f^\circ}{(\delta x/\ell)} = 2^m (m+1) H^m(x) [1 - S] \tag{22}$$

If all of the specimens are tested to failure (i.e., $S = 0$)

$$\frac{\delta f^\circ}{(\delta x/\ell)} = 2^m (m+1) H^m(x) \tag{23}$$

which is illustrated in Fig. 1.

As an example of a coating application, consider the segment size of coatings after fabrication. Often after fabrication, coatings will form a mud-flat cracking pattern. For the purposes of the analysis, assume the following: 1) the coating breaks into hexagonal segments of side length a and 2) the stress state is driven by the substrate and due to the thermal mismatch between the substrate and the coating, then

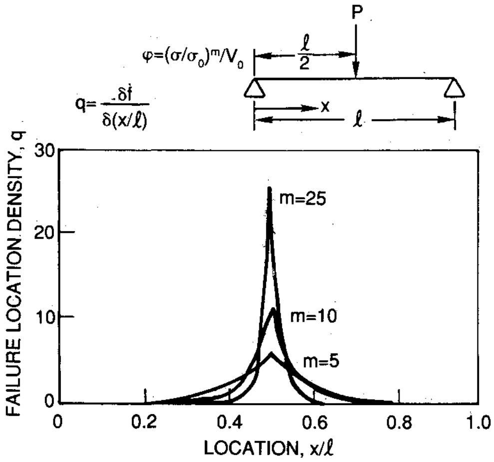

Fig. 1 Three-point bending failure location (taken from Ref. 1).

$$\sigma_x = \sigma_y = \frac{\alpha E \Delta T}{1 - \nu} \tag{24}$$

where σ_x and σ_y are the in-plane stresses, α the difference in the coefficients of thermal expansion, E and ν the elastic constants, ΔT the drop in temperature from the glass transition temperature to room temperature, and σ_x and σ_y the in-plane normal stresses, and all other stresses vanish.

Then using the multiaxial failure model, Eq. (4), for an isotropic material and applying Eq. (14) gives for the survival probability

$$1 - F = S = \exp\left\{ -\frac{3\sqrt{3}}{2}\left(\frac{a^2 h}{V_o}\right)\left[\frac{\sqrt{2}\alpha E(\Delta T)}{(1-\nu)\sigma_o}\right]^m \right\} \tag{25}$$

where h is the coating thickness. The probability of the segment size to be between a and $a + \mathrm{d}a$ is $\mathrm{d}F$ and

$$\frac{\mathrm{d}F}{\mathrm{d}(ka)} = (ka)e^{-\frac{1}{2}(ka)^2} \tag{26}$$

where

$$k^2 = 3\sqrt{3}\left(\frac{h}{V_o}\right)\left[\frac{\sqrt{2}\alpha E(\Delta T)}{(1-\nu)\sigma_o}\right] \tag{27}$$

The probability distributions are illustrated in Fig. 2.

Time-Dependent Failure

If failure occurs randomly over the volume (i.e., space), as well as time, then the infinitesimal probability for the failure in the infinitesimal volume δV, and time δt, should be represented by

$$\delta f = \dot{\psi} \delta V \delta t \tag{28}$$

where

$$(\dot{\ }) = \frac{\partial()}{\partial t}$$

The parameter $\dot{\psi}$ is again a function of the state of the material in δV at time t and will be shown to be a time derivative. The function, though, will not be the time derivative of ψ in Eq. (1). Then assuming any point failing at any time causes the entire structure to fail (i.e., weakest link theory is assumed to hold), the probability for survival S is given by the product of all of the probabilities for survival

$$S = \prod_{k,l}(1 - \delta f_{k,l}) = \prod_{k,l} \delta_{S_{k,l}} \tag{29}$$

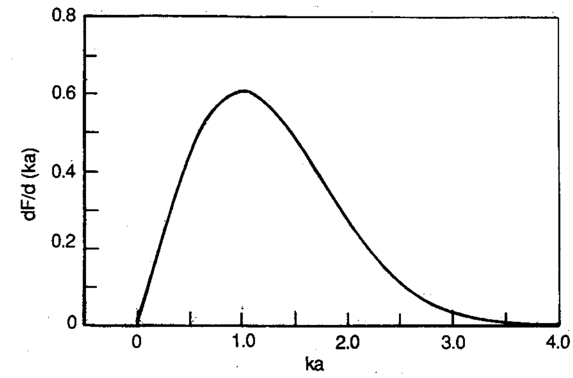

Fig. 2 Segment size density distribution.

where $\delta_{S_{k,l}}$ is the probability for survival for element δV_k over time interval δ_{t_l}.

Substituting Eq. (28) into Eq. (29) gives

$$S = \prod_{k,l}(1 - \dot{\psi}\delta V_k \delta t_l) \tag{30}$$

Taking the natural logarithm of Eq. (30) turns the product into a sum, and

$$\ln S = \sum_{k,l} \ln\left(1 - \dot{\psi}\delta V_k \delta t_l\right) \tag{31}$$

Since

$$\ln(1 + x) \approx x$$

for small x, Eq. (31) is

$$\ln S = -\sum_{k,l} \dot{\psi}\delta V_k \delta t_l \tag{32}$$

As the volumes and time intervals each approach zero, the sum becomes an integral, or

$$\ln S = -\int_V \int_0^t \dot{\psi}\,\mathrm{d}t\,\mathrm{d}V \tag{33}$$

which is exactly analogous to Eq. (12). Then,

$$S = \exp\left\{-\int_V \int_0^t \dot{\psi}\,\mathrm{d}t\,\mathrm{d}V\right\} \tag{34}$$

The determination of the failure distribution densities can be found by noting that, in order to observe a failure in interval δt and volume δV, the structure must have survived to time t. Then,

$$\delta f^\circ = S\delta f = S\dot{\psi}\delta V \delta t \tag{35}$$

where δf° is the infinitesimal probability to observe a failure in interval δt and volume δ_V. Integrating over time

$$\frac{\delta f^\circ}{\delta V} = \int_0^t S\dot{\psi}\,\mathrm{d}t \tag{36}$$

Several approaches can be used to find functional forms for $\dot{\psi}$ in Eq. (28). They include the following: 1) simply taking the parameters in the distribution to be time (or cycle) dependent,[11] 2) using existing damage models,[6] 3) using fatigue or creep rupture data directly. In the first choice, the characteristic strength σ_o in Eq. (12) in a Weibull distribution could be made time dependent. If the exponent m in Eq. (11) is also time dependent, then probabilities for survival would not always shrink. Hence, only the characteristic strength can change with time. For this case, the equations for static failure can be used directly. Then,

$$S(t) = \exp\left\{-\int_V \left[\frac{\sigma(t)}{\sigma_o(t)}\right]^m \frac{\mathrm{d}V}{V_o}\right\} \tag{37}$$

and

$$\frac{\delta f^{\circ}}{\delta V} = \int_0^t S(t')\left\{ m\left[\frac{\sigma(t')}{\sigma_o(t')}\right]^{m-1}\dot{\sigma}(t')\mathrm{d}t'\right\} \tag{38}$$

This has been applied to experimental components and found to be accurate.[11]

The second option is to apply damage theories. As an example, the damage theory of Chabache will be used, as presented in Ref. 6. In this theory, a damage parameter D is assumed to vary between zero where there is no accumulated damage and 1 when failure at a point has occurred. Both creep damage D_c and fatigue damage D_f are considered, and

$$D = D_c + D_f \tag{39}$$

The creep damage is governed by

$$\dot{D}_c = A\left[\frac{\sqrt{3J_2}}{1-D_c}\right]^r (1-D_c)^{r-k} \tag{40}$$

$$k = k_o \exp\left(\mu\sqrt{3j_2}\right) \tag{41}$$

where $\dot{D}_c = \mathrm{d}D_c/\mathrm{d}t$, A, r, k_o, and μ are material parameters, and J_2 is the second invariant of the deviatoric stress tensor. For a uniaxial stress σ,

$$J_2 = \frac{1}{3}\sigma^2 \tag{42}$$

The fatigue damage grows per cycle N according to

$$\frac{\mathrm{d}D_f}{\mathrm{d}N} = [1-(1-D_f)^{\beta+1}]^a\left[\frac{\Delta\sigma_{II}}{M_o(1-b\sigma_I)(1-D_f)}\right]^{\beta} \tag{43}$$

where a, β, M_o, and b are material parameters, which may depend on $\Delta\sigma_{II}$ abd σ_I,

$$\sigma_I = \left[\left(\frac{\sigma_1^{\max}+\sigma_1^{\min}}{2}\right)+\left(\frac{\sigma_2^{\max}+\sigma_2^{\min}}{2}\right)+\left(\frac{\sigma_3^{\max}+\sigma_3^{\min}}{2}\right)\right] \tag{44}$$

$$\Delta\sigma_{II} = \left[\frac{\left\{\left(\sigma_1^{\max}-\sigma_1^{\min}\right)-\left(\sigma_2^{\max}-\sigma_2^{\min}\right)\right\}^2}{2} + \frac{\left\{\left(\sigma_2^{\max}-\sigma_2^{\min}\right)-\left(\sigma_3^{\max}-{}_3^{\min}\right)\right\}^2}{2} + \frac{\left\{\left(\sigma_3^{\max}-\sigma_3^{\min}\right)-\left(\sigma_1^{\max}-\sigma_1^{\min}\right\}^2\right.}{2}\right]^{1/2} \tag{45}$$

where $\sigma_i^{\max}$ and $\sigma_i^{\min}$ are the maximum and minimum ith principal stress values during each fatigue cycle, respectively.

A straightforward conversion can be accomplished by taking

$$D_c = 1 - e^{-a_c\psi_c} \tag{46}$$

$$D_f = 1 - e^{-a_f\psi_f} \tag{47}$$

where $\psi = \psi_f + \psi_c + \psi_s$, and ψ_s represents static failure.

Equations (46) and (47) can be substituted directly into Eq. (40) and (43) where the time derivatives are simply

$$\dot{D}_c = a_c\dot{\psi}_c e^{-a_c\psi_c} \tag{48}$$

$$\frac{\mathrm{d}D_f}{\mathrm{d}N} = a_f\dot{\psi}_f e^{-a_f\psi_f}\bigg/\left(\frac{\mathrm{d}N}{\mathrm{d}t}\right) \tag{49}$$

Since $N = 2\pi\omega t$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 2\pi\omega$$

which is just the frequency in cycles per unit time. The creep and fatigue failure parameters can be taken as

$$\dot{\psi}_c = V_o f_c(\psi) \tag{50}$$

$$\dot{\psi}_f = V_o f_f(\psi)\dot{W} \tag{51}$$

where

$$f_c(\psi) = \frac{a_c(V_o\psi)^r}{1+a_1\exp\{b_1\psi^{1/s}\}} \tag{52}$$

$$f_f(\psi) = a_f(1-e^{(1+b_2)b\psi})^{p_1}\left\{\frac{\psi^{g/s}}{e^{b\psi}(1-b_o\psi)^{1/s}}\right\} \tag{53}$$

$$p_1 = 1 - \frac{P_o}{C_1(1-b_2)}\psi^{C_2/s} \tag{54}$$

$$\dot{W} = \begin{cases}\sigma_{ij}\dot{\epsilon}_{ij} & \sigma_{ij}\dot{\epsilon}_{ij} > 0\\ 0 & \sigma_{ij}\epsilon_{ij} < 0\end{cases}$$

σ_{ij} and ϵ_{ij} are the stress and strain tensors.

For a uniaxial stress state,

$$\dot{W} = \begin{cases}\sigma\dot{\epsilon} & \sigma\dot{\epsilon} > 0\\ 0 & \sigma\dot{\epsilon} \le 0\end{cases}$$

The total failure parameter is now given by summing the fatigue, creep, and static parts. Instead of damage, entropy gain[12] could be used in a similar manner. Using damage, or entropy gain, has the disadvantage of requiring many parameters. The parameters in Eqs. (51–54) must be found by matching experimental data.

The third approach offers the best compromise in terms of simplicity and completeness by using existing fatigue or creep rupture directly. In this approach, it is more difficult to derive the equations for the failure parameter ψ.

In the modeling of fatigue failure, it is commonly assumed that the number of cycles to failure N and the maximum applied cyclic stress σ_m are related by[12]

$$N\sigma_m^a = \text{const} \tag{55}$$

If the constant in Eq. (55) is assumed to be a function of the survival probability S_N, then after N cycles

$$CN^n\sigma_m^m = k(S_N) \tag{56}$$

where $m = na$ and C is a constant. The constant can be taken so that

$$\left(\frac{V}{V_o}\right)\left(\frac{N}{N_o}\right)^n\left(\frac{\sigma_m}{\sigma_f}\right)^m = k(S_N) \tag{57}$$

where $V_oN_o^n\sigma_f^m$ is a material parameter.

Solving Eq. (57) for S_N, and assuming a uniform stress state

$$S_N = e^{-\psi_N V} = k^{-1}\left[\left(\frac{V}{V_o}\right)\left(\frac{N}{N_o}\right)^n\left(\frac{\sigma_m}{\sigma_f}\right)^m\right] \tag{58}$$

Taking

$$k^{-1}(x) = e^{-x}$$

Eq. (58) becomes

$$\psi_N = \frac{1}{V_o}\left(\frac{N}{N_o}\right)^n\left(\frac{\sigma_m}{\sigma_f}\right)^m \tag{59}$$

In order to arrive at a differential form for ψ, we must consider the change in the failure parameter over one cycle

$$\psi_{N+1} - \psi_N = \Delta\psi_N = \frac{1}{V_o}\left[\left(\frac{N+1}{N_o}\right)^n - \left(\frac{N}{N_o}\right)^n\right]\left(\frac{\sigma_m}{\sigma_f}\right)^m \quad (60)$$

For high cycle fatigue, we only need to consider the case where

$$N >> 1$$

then

$$\Delta\psi_N = \frac{n}{V_o N_o}\left(\frac{N}{N_o}\right)^{n-1}\left(\frac{\sigma_m}{\sigma_f}\right)^m \quad (61)$$

This is equivalent to an integration over the cycle while the stress is positive,

$$\Delta\psi_N = \int_{\sigma=0}^{\sigma=\sigma_m} d\psi_f = \frac{nm}{V_o N_o \sigma_o}\left(\frac{N}{N_o}\right)^{n-1}\int_{\sigma=0}^{\sigma=\sigma_m}\left(\frac{\sigma}{\sigma_f}\right)^{m-1} d\sigma \quad (62)$$

Equating the differentials in Eq. (62) gives

$$\dot\psi_f = \frac{mn}{V_o N_o}\left(\frac{N}{N_o}\right)^{n-1}\left(\frac{\sigma}{\sigma_f}\right)^{m-1}\frac{\dot\sigma}{\sigma_f} \quad (63)$$

But from Eq. (59)

$$\frac{N}{N_o} = \left[\frac{V_o\psi_N}{(\sigma_m/\sigma_f)^m}\right]^{1/n} = \left[\frac{V_o\psi_f}{(\sigma_m/\sigma_f)^m}\right]^{1/n} \quad (64)$$

In Eq. (64), the maximum stress can be replaced by the current stress if a multiplicative constant K is introduced, which will be determined later. Then if $\Delta\psi_N$ is much less than ψ_N we can set

$$\frac{N}{N_o} = K\left[\frac{V_o\psi_f}{(\sigma/\sigma_f)^m}\right]^{1/n} \quad (65)$$

substituting Eq. (65) into Eq. (63) gives

$$\dot\psi_f = \frac{mnK}{V_o N_o}\left(V_o\psi_f\right)^{1-\frac{1}{n}}\left\langle\frac{\sigma}{\sigma_f}\right\rangle^{m-1}\left\langle\frac{\dot\sigma}{\sigma_f}\right\rangle \quad (66)$$

Equation (66) will be taken to be the equation governing the growth in the failure parameter, which only occurs when both σ and $\dot\sigma$ are greater than zero. Recall $\langle x\rangle$ is the unit ramp function. Equation (66) can be integrated over one cycle with the initial condition

$$\psi_f = \psi_N \quad \text{at} \quad \sigma = 0$$

then

$$\psi_{N+1} = \frac{a}{V_o}\left[(V_o\psi_N)^{1/n} + \frac{Kn}{N_o}\left(\frac{\sigma}{\sigma_f}\right)^{m/n}\right]^n \quad (67)$$

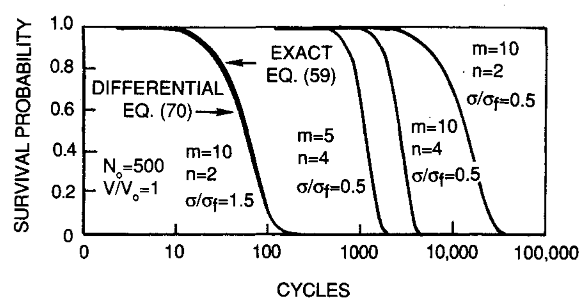

Fig. 3 Fatigue survival probability using exact and differential forms.

Taking

$$K = 1/n \quad (68)$$

will give the correct answer for $N = 0$.

Then Eqs. (66) and (67) become

$$\dot\psi_f = \frac{m}{V_o N_o}(V_o\psi_f)^{\left(1-\frac{1}{n}\right)}\left\langle\frac{\sigma}{\sigma_f}\right\rangle^{\frac{m}{n}-1}\left\langle\frac{\dot\sigma}{\sigma_f}\right\rangle \quad (69)$$

$$\psi_{N+1} = \frac{1}{V_o}\left[(V_o\psi_N)^{1/n} + \frac{1}{N_o}\left(\frac{\sigma_m}{\sigma_f}\right)^{m/n}\right]^n \quad (70)$$

In Fig. 3, Eqs. (59) and (70) are compared. The results are seen to be nearly identical, even for a small number of cycles. Figure 4 compares the uniaxial, Eq. (14), pure bending, Eq. (19), and three point bending load cases, Eq. (21). For the same maximum stresses, the life can vary by an order of magnitude in these three load cases.

A probabilistic failure model for creep rupture can be developed in an analogous manner. Taking

$$\psi_c = \frac{1}{V_o}\left(\frac{\sigma}{\sigma_c}\right)^M\left(\frac{t}{t_o}\right)^N \quad (71)$$

where the stress is now considered a constant.

Then

$$\dot\psi_c = \frac{N}{V_o t_o}\left(\frac{\sigma}{\sigma_c}\right)^M\left(\frac{t}{t_o}\right)^{N-1} \quad (72)$$

From Eq. (71)

$$\left(\frac{t}{t_o}\right) = (V_o\psi_c)^{1/N}\left(\frac{\sigma}{\sigma_c}\right)^M \quad (73)$$

Substituting Eq. (73) into Eq. (72) and assuming that only positive stresses cause failure

$$\dot\psi_c = \frac{N}{V_o t_o}\left\langle\frac{\sigma}{\sigma_c}\right\rangle^{M/N}(V_o\psi_c)^{1-\frac{1}{N}} \quad (74)$$

The failure parameter ψ can now be taken as the sum of Eq. (69) and (74). Static failure can be represented by the maximum value of the static failure parameter over all times, or

$$\psi_s = \max_t(\psi_o) \quad (75)$$

where $G(\sigma_i, \psi_o) = 0$ from Eqs. (2) and (3).

The total ψ is

$$\psi = \psi_s + \int_o^t\left[\dot\psi_f(t') + \dot\psi_c(t')\right]dt' \quad (76)$$

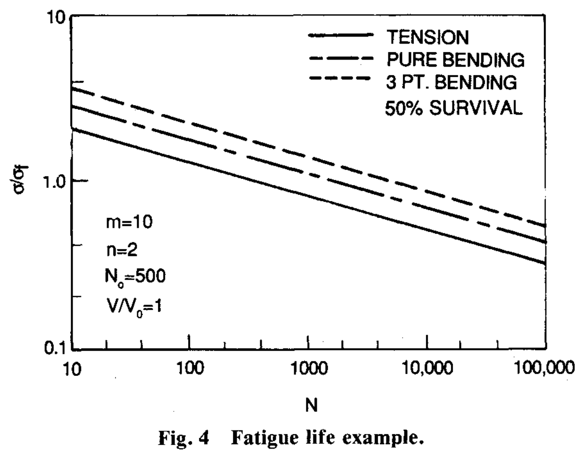

Fig. 4 Fatigue life example.

The evolution Eqs. (74) and (69) can be written as

$$\dot{\psi}_c = \frac{N}{V_o t_o}\left\langle\frac{\sigma}{\sigma_c}\right\rangle^{m/N}(V_o\psi)^{1-\frac{1}{N}} \tag{77}$$

$$\dot{\psi}_f = \frac{m}{V_o N_o}\left\langle\frac{\sigma}{\sigma_f}\right\rangle^{\frac{m}{n}-1}(V_o\psi)^{1-\frac{1}{n}}\left\langle\frac{\dot{\sigma}}{\sigma_f}\right\rangle \tag{78}$$

respectively. The right-hand sides of Eqs. (77) and (78) now contain the total failure parameter instead of the individual creep and fatigue failure parameters. This should be more accurate if both creep and fatigue failure are driven by the same mechanism (e.g., the propagation of cracks along grain boundaries). Equations (75–78) contain the complete uniaxial stress state model that will be applied in the next section. Multiaxial stress states can be described by expressions similar to Eq. (3) and will be briefly examined in the next section.

Sample Applications

As an example, consider a coating driven by a much stiffer substrate through a strain and thermal cycle. Consider first a one-dimensional case. The coating will be modeled using a glass transition temperature. Above the glass transition temperature the coating cannot support any stresses, and below the glass transition temperature the coating is elastic. The coating strain will be taken to be that of the substrate. Hooke's law, in one dimension, gives for the coating stress

$$\sigma = E\left[\epsilon - \alpha(T - T_r)\right] \tag{79}$$

where σ is the coating stress; E the Young's Modulus of the coating; α the coefficient of thermal expansion of the coating; T the coating temperature; ϵ the coating strain, which is equal to the substrate strain; and T_r a reference temperature.

The coating reference temperature can be found by noting that, at the glass transition temperature, the stress vanishes. Then

$$\epsilon_g - \alpha\left(T_g - T_R\right) = 0 \tag{80}$$

where ϵ_g is the applied substrate strain when the temperature was last equal to the glass transition temperature T_g. Solving Eq. (80) for T_R gives

$$T_R = T_g - \left(\epsilon_g/\alpha\right) \tag{81}$$

Substituting Eq. (81) into Eq. (79) completes the stress strain law

$$\sigma = \begin{cases} E[\epsilon - \epsilon_g - \alpha(T - T_g)] & T < T_g \\ 0 & T \geq T_g \end{cases} \tag{82}$$

The applied strain and temperature history will be taken as triangular waves, as shown in Figs. 5 and 6. The corresponding stress history is shown in Fig. 7 for the data in Table 1. The stress history can be integrated over one cycle according to Eq. (69) to yield the evolution of the failure parameter ψ and the corresponding survival probability as

$$\psi_{N+1} = \frac{1}{V_o}\left\{(V_o\psi_N)^{1/N} + \frac{m}{V_o N_o}\delta\right\}^n \tag{83}$$

$$S_{N+1} = e^{-\psi_{N+1}V} \tag{84}$$

where

$$\delta = \int_{\text{cycle}}\left(\frac{<\sigma>}{\sigma_f}\right)^{\frac{m}{n}-1}\left\langle\frac{\dot{\sigma}}{\sigma_f}\right\rangle \mathrm{d}t \tag{85}$$

The probability of survival as a function of cycle number for the parameters of Table 1, and the case where V and V_o are the same, is shown in Fig. 8. As a multiaxial example, consider a biaxial stress state in a coating that lies in the $x-y$

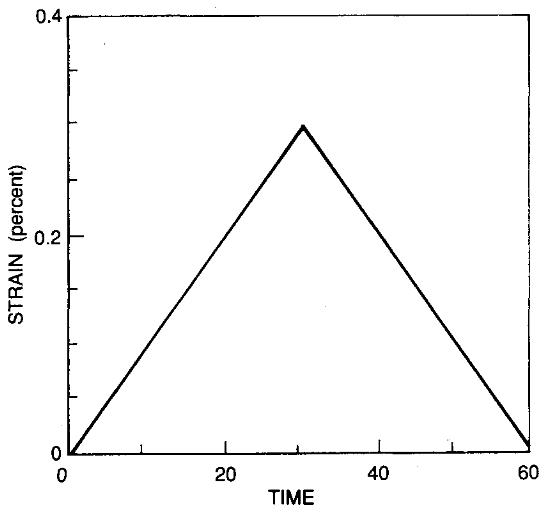

Fig. 5 Strain history for coating.

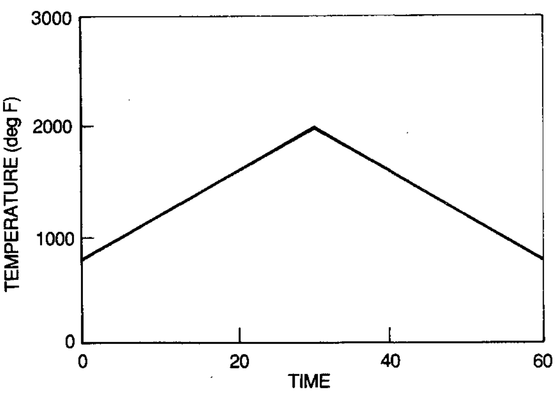

Fig. 6 Temperature history for coating.

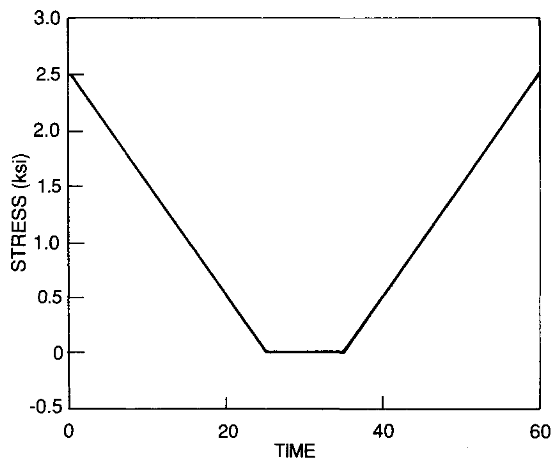

Fig. 7 Coating stress history.

Table 1 Values of parameters for coating example

Parameter	Value
m	10
n	2
N_o	500
σ_f	5000 psi
α	$5 \times 10^{-6}/°F$
T_g	1800 F
E	10^6 psi

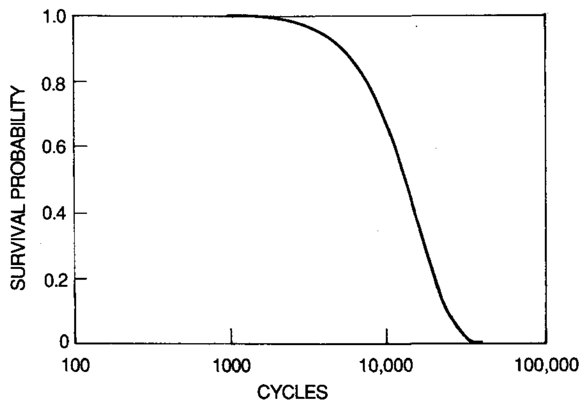

Fig. 8 Survival probability history of coating.

plane. Hooke's law is

$$\epsilon_x = \left(1/E\right)\left(\sigma_x - \nu\sigma_y\right) + \alpha\left(T - T_g\right) + \epsilon_{x_g} \tag{86}$$

$$\epsilon_y = \left(1/E\right)\left(\sigma_y - \nu\sigma_x\right) + \alpha\left(T - T_g\right) + \epsilon_{y_g} \tag{87}$$

$$\gamma_{xy} = \left(1/G\right)\tau_{xy} + \gamma_{xy_g} \tag{88}$$

where G is the shear modulus, γ_{xy} the coating shear strain, τ_{xy} the coating shear stress, and γ_{xy_g} the applied substrate strain at the glass transition temperature.

A multiaxial law is now required for the failure parameter. This can now be taken, in a manner similar to Eq. (3), as

$$\dot{\psi} = \frac{m}{V_o N_o}(V_o \psi)^{\left(1-\frac{1}{n}\right)} \left\{ \left(\frac{<\sigma_x>}{\sigma_{x_f}}\right)^p \left(\frac{<\dot{\sigma}>}{\sigma_{x_f}}\right)^{p/m} + \left(\frac{<\sigma_y>}{\sigma_{x_f}}\right)^p \left(\frac{<\dot{\sigma}_y>}{\sigma_{x_f}}\right)^{p/m} \left(\frac{|\tau_{xy}|}{\tau_f}\right)^p \left(\frac{|\dot{\tau}_{xy}|}{\tau_f}\right)^{p/m} \right\}^{m/p} \tag{89}$$

For the case where the imposed strains are isotropic

$$\epsilon_x = \epsilon_y = \epsilon \tag{90}$$

$$\gamma_{xy} = 0 \tag{91}$$

Then

$$\sigma_x = \sigma_y = \sigma \tag{92}$$

$$\tau_{xy} = 0 \tag{93}$$

Since the properties are isotropic, Eq. (89) becomes

$$\dot{\psi} = 2^{m/p}\left(\frac{M}{V_o N_o}\right)(V_o \psi)^{\left(1-\frac{1}{n}\right)}\left(\frac{<\sigma>}{\sigma_f}\right)^m \frac{<\dot{\sigma}>}{\sigma_f} \tag{94}$$

where

$$\sigma_{x_f} = \sigma_{y_f} = \sigma_f \tag{95}$$

Equation (94) is equivalent to Eq. (78) to within a multiplicative constant. Equation (89) can use the principal stresses instead of the actual stress components. Many other forms also need to be examined. For anisotropic materials, more complicated expressions need to be developed.

The model has been added as a postprocessor to a finite element code and has been used to analyze design concepts for an advanced gas turbine engine combustor.[11] The applications have described accurately the results of experimental tests, but in each case, static failure has dominated and, hence, a real test of the model still needs to be done.

Conclusions

Probabilistic formulations for static failure have been extended to time-dependent failure. Three methods for completing the extension have been described. An example for a multiaxial formulation has been given and an application to uniaxial stress states has demonstrated the results that can be obtained. The model still needs a consistent multiaxial formulation especially for anisotropic materials and needs to be compared to a wider variety of experimental results.

Appendix: Sum of Failure and Survival Probabilities

From Eq. (6), the survival probability is

$$S = \exp[-\int \psi \, \mathrm{d}V] \tag{A1}$$

and from Eq. (9), the failure probability is

$$F = \int_V \frac{\delta f^\circ}{\delta V} \mathrm{d}V \tag{A2}$$

where from Eq. (7)

$$\frac{\delta f^\circ}{\delta V} = \int_o^t S(t_1)\psi,_{t_1}(\boldsymbol{x}, t_1)\mathrm{d}t_1 \tag{A3}$$

where

$$\psi,_t = \frac{\partial \psi}{\partial t}$$

Let

$$P(t) = F(t) + S(t) \tag{A4}$$

then

$$P(t) = \int_V \int_o^t \exp\,[-\int_V \psi \, \mathrm{d}V]\psi,_t \, \mathrm{d}t_1 \, \mathrm{d}V + \exp - [\int_V \psi \, \mathrm{d}V] \tag{A5}$$

Differentiating Eq. (A-5)

$$\frac{\partial P}{\partial t} = \int_V \exp[-\int_V \psi \, \mathrm{d}V]\psi,_t \, \mathrm{d}V - \exp[-\int_V \psi,_t \, \mathrm{d}V] \tag{A6}$$

But

$$\exp\,[-\int_V \psi \, \mathrm{d}V] = S(t)$$

from Equation (A1) and, hence,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0 \tag{A7}$$

Then integrating

$$P = C_1 = F + S \tag{A8}$$

where C_1 is an arbitrary constant.

If there are no failures due to residual stresses resulting from fabrication, then

$$S = 1 \quad \text{and} \quad F = 0 \quad \text{at } t = 0 \tag{A9}$$

the arbitrary constant is then 1 and Equation (A-8) becomes

$$F + S = 1 \tag{A10}$$

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